

Lower Multiplicity for Irreducible Representations of Nilpotent Locally Compact Groups

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Let G be a nilpotent locally compact group. The lower multiplicity $M_L(\pi)$ is defined for every irreducible representation π of G , which does not form an



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1. INTRODUCTION AND RESULTS

The upper and lower multiplicities, $M_U(\pi)$ and $M_L(\pi)$, were introduced in [1] for irreducible representations π of an arbitrary C^* -algebra A in order to study the properties of trace functions $\pi \rightarrow \text{tr } \pi(a)$ on the dual space \hat{A} of A . On the one hand they appear as multiplicity numbers in formulae for limits of traces [4, Theorem 4.1]. On the other hand, $M_U(\pi)$ and $M_L(\pi)$ correspond to the maximal and the minimal count, respectively, of the number of nets of orthogonal equivalent pure states which can simultaneously converge to a given pure state associated to π , and hence these numbers reflect the strength of convergence in \hat{A} . In addition, since $M_U(\pi) = 1$ if and only if π satisfies Fell's condition [1, Theorem 4.1], the gap between $M_U(\pi)$ and $M_L(\pi)$ may also be viewed as a measure of the extent to which Fell's condition fails for π .

Using the main result of [16], it has been shown in [4, Corollary 2.9] that if G is a connected and simply connected nilpotent Lie group and $\pi \in \hat{G}$, then $M_U(\pi) < \infty$ if and only if the Kirillov orbit associated to π has maximal dimension. Ludwig [15] has given an example of a simply connected nilpotent Lie group G and $\pi \in \hat{G}$ for which $1 < M_U(\pi) < \infty$. In subsequent work with R. J. Archbold, J. Ludwig, G. Schlichting, and D. W. B. Somerset the upper multiplicity will be investigated in more detail for simply connected nilpotent Lie groups. In a different direction explicit



formulae were recently obtained for both $M_U(\pi)$ and $M_L(\pi)$ for irreducible representations of Moore groups (that is, groups with finite dimensional irreducible representations) [3]. In particular, it follows that, within the class of almost abelian discrete groups, $M_U(\pi)$ attains all integer values.

In this paper we study the lower multiplicity for nilpotent locally compact groups. Motivated by low-dimensional examples, the required data for which can be found in [19], R. J. Archbold conjectured that $M_L(\pi) = 1$ for every irreducible representation π of a simply connected nilpotent Lie group. Our first purpose is to verify the following slightly more general version of that conjecture.

THEOREM 1. *Let G be a connected nilpotent group and let π be an irreducible representation of G . If $\{\pi\}$ is not open in \hat{G} , then $M_L(\pi) = 1$.*

Recall that $M_L(\pi)$ is not defined whenever the singleton $\{\pi\}$ is open in \hat{G} [1, p. 123]. Therefore in the following theorems the hypothesis that G be noncompact is necessary. We next show that the conclusion of Theorem 1 remains true for arbitrary nilpotent groups if we impose a strong condition on π .

THEOREM 2. *Let G be a non-compact nilpotent locally compact group. Then $M_L(\pi) = 1$ for every finite dimensional irreducible representation π of G .*

For G the discrete Heisenberg group, it is easy to see that $M_L(\pi) = \infty$ for every infinite dimensional $\pi \in \hat{G}$. Our final result is a far reaching generalization of that particular example. Recall that a locally compact group G is called SIN-group (group with small invariant neighbourhoods) if G has a neighbourhood basis of the identity consisting of sets V such that $xVx^{-1} = V$ for all $x \in G$. Of course, this class comprises all discrete groups and all groups with open centre.

THEOREM 3. *Let G be a non-compact amenable SIN-group with T_1 primitive ideal space $\text{Prim}(C^*(G))$. Then \hat{G} has no open points and so $M_L(\pi)$ is defined for every irreducible representation π of G . Moreover, if $M_L(\pi) < \infty$ then π is finite dimensional.*

Since $\text{Prim}(C^*(G))$ is a T_1 space for every nilpotent SIN-group G , we can deduce the following corollary from Theorem 2 and 3.

COROLLARY. *Let G be a non-compact nilpotent SIN-group. Then, for every irreducible representation π of G , $M_L(\pi)$ is defined and either $M_L(\pi) = 1$ or $M_L(\pi) = \infty$. Moreover, $M_L(\pi) = 1$ if and only if π is finite dimensional.*

It is possible to construct examples of C^* -algebras and irreducible representations with finite lower multiplicity greater than one (see [1]). The preceding results, however, raise the question of whether the situation $1 < M_L(\pi) < \infty$ can ever occur in the dual of a locally compact group.

2. PRELIMINARIES

Let A be a C^* -algebra and let π be an irreducible representation of A . We refrain from recalling the definitions of upper and lower multiplicity $M_U(\pi)$ and $M_L(\pi)$ of π and instead refer the reader to [1, 4]. Similarly, one can define upper and lower multiplicities relative to a net Ω in the dual space \hat{A} of A : $M_U(\pi, \Omega)$ and $M_L(\pi, \Omega)$. In establishing the results of this paper we shall have to employ various results about multiplicities from [1, 3, 4] rather than formal definitions.

For $\pi \in \hat{A}$, $M_L(\pi)$ is defined only when $\{\pi\}$ is not open in \hat{A} . Now let $\Omega = (\pi_\alpha)_\alpha$ be a net in \hat{A} converging to π such that eventually $\pi_\alpha \neq \pi$. Then

$$M_L(\pi) \leq M_L(\pi, \Omega) \leq M_U(\pi, \Omega) \leq M_U(\pi)$$

[2, Proposition 2.1]. There are nets Ω_1 and Ω_2 such that $M_L(\pi, \Omega_1) = M_L(\pi)$ and $M_U(\pi, \Omega_2) = M_U(\pi)$. The property that $M_U(\pi) < \infty$ for all $\pi \in \hat{A}$ is equivalent to A being a bounded trace C^* -algebra [4, Theorem 2.6]. As mentioned earlier, for fixed $\pi \in \hat{A}$, $M_U(\pi) = 1$ if and only if π satisfies Fell's condition (that is, there exist a neighbourhood V of π in \hat{A} and $a \in A^+$ such that $\rho(a)$ is a rank one projection for all $\rho \in V$) [1, Theorem 4.6]. In particular, if A is a continuous trace C^* -algebra, then $M_U(\pi) = 1$ for all $\pi \in \hat{A}$. Much more general, Theorem 4.1 of [4] shows how multiplicities are related to convergence of traces of irreducible representations. Let $\Omega = (\pi_\alpha)_\alpha$ be a net in \hat{A} and $F \subseteq \hat{A}$, and suppose that there exist $m_\pi \in \mathbb{N}(\pi \in F)$ and a dense $*$ -subalgebra B of A such that

$$\operatorname{tr} \pi_\alpha(a) \rightarrow \sum_{\pi \in F} m_\pi \operatorname{tr} \pi(a) < \infty$$

for all $a \in B^+$. Then $m_\pi = M_L(\pi, \Omega) = M_U(\pi, \Omega)$ for each $\pi \in F$.

We now turn to locally compact groups and group C^* -algebras and introduce some notation and basic facts that will be used throughout the paper. Let G be a locally compact group and $C^*(G)$ the group C^* -algebra of G . As is customary, we shall use the same letter, for example π , to denote a unitary representation of G and the associated $*$ -representation of $C^*(G)$. Then $\ker \pi$ will denote the C^* -kernel of π , and $\pi \rightarrow \ker \pi$ defines a mapping from the dual space \hat{G} of G onto $\operatorname{Prim}(C^*(G))$, the primitive ideal space of $C^*(G)$. If S and T are sets of unitary representations of G , then S is weakly

contained in $T(S < T)$ if $\bigcap_{\sigma \in S} \ker \sigma \supseteq \bigcap_{\tau \in T} \ker \tau$, and S and T are weakly equivalent ($S \sim T$) if $S < T$ and $T < S$ (see [6, 7]). The topology on \hat{G} is the pullback of the hull-kernel topology on $\text{Prim}(C^*(G))$. Thus, for $S \subseteq \hat{G}$ and $\pi \in \hat{G}$, π belongs to the closure of S if and only if $\pi < S$.

For a closed subgroup H of G and unitary representations π of G and τ of H , $\pi|_H$ denotes the restriction of π to H and $\text{ind}_H^G \tau$ the representation of G induced by τ . We have the tensor product formula $\pi \otimes \text{ind}_H^G \tau = \text{ind}_H^G(\pi|_H \otimes \tau)$.

3. PROOF OF THEOREM 1

Let G be a connected and simply connected nilpotent Lie group with Lie algebra \mathfrak{g} . Kirillov's theory gives a bijection between \hat{G} and $\mathfrak{g}^*/\text{Ad}^*$, the orbit space of the coadjoint representation of G on the dual vector space \mathfrak{g}^* . Indeed, each $f \in \mathfrak{g}^*$ gives rise to an irreducible representation π_f of G , and π_f is unitarily equivalent to π_g , $g \in \mathfrak{g}^*$, if and only if $g \in \text{Ad}^*(G)f$ (see, for example, [5]). The Kirillov correspondence $\text{Ad}^*(G)f \rightarrow \pi_f$ is continuous (in fact, a homeomorphism) provided that $\mathfrak{g}^*/\text{Ad}^*$ carries the quotient topology.

LEMMA 1. *Let \mathfrak{g} be a nilpotent Lie algebra with centre \mathfrak{z} and let $G = \exp \mathfrak{g}$. Suppose that $f \in \mathfrak{g}^*$ is such that $\text{Ad}^*(G)f \neq f + \mathfrak{z}^\perp$. Then there exist ideals \mathfrak{a} and \mathfrak{b} of \mathfrak{g} and a sequence $(f_n)_n$ in \mathfrak{g}^* with the following properties:*

- (i) $\mathfrak{z} \subseteq \mathfrak{b} \subseteq \mathfrak{a}$, $\dim(\mathfrak{a}/\mathfrak{b}) = 1$ and $\mathfrak{a}/\mathfrak{b}$ is contained in the centre of $\mathfrak{g}/\mathfrak{b}$.
- (ii) $\text{Ad}^*(G)f = \text{Ad}^*(G)f + \mathfrak{a}^\perp$, $f_n \rightarrow f$ in \mathfrak{g}^* and, for all $n \in \mathbb{N}$, $f_n|_{\mathfrak{b}} = f|_{\mathfrak{b}}$, $f_n \notin \text{Ad}^*(G)f$ and $\text{Ad}^*(G)f_n = \text{Ad}^*(G)f_n + \mathfrak{a}^\perp$.

Proof. Let X_1, \dots, X_m be a strong Malcev basis of \mathfrak{g} through the ascending central series of \mathfrak{g} and let X_1^*, \dots, X_m^* denote the dual basis of \mathfrak{g}^* . Let d be minimal such that

$$\text{Ad}^*(G)f = \text{Ad}^*(G)f + \sum_{j=d+1}^m \mathbb{R}X_j^*.$$

Set $\mathfrak{a} = \sum_{j=1}^d \mathbb{R}X_j$ and $\mathfrak{b} = \sum_{j=1}^{d-1} \mathbb{R}X_j$. The hypothesis on f implies that $\mathfrak{z} \subseteq \mathfrak{b}$, and hence (i) and the first property in (ii) are satisfied.

Let T be the set of all $t \in \mathbb{R}$ such that $f + tX_d^* \in \text{Ad}^*(G)f$. T is a subgroup of \mathbb{R} . Indeed, if $t_1, t_2 \in \mathbb{R}$ and $x_1, x_2 \in G$ are such that $\text{Ad}^*(x_i)f = f + t_iX_d^*$, $i = 1, 2$, then

$$\begin{aligned} \text{Ad}^*(x_1)\text{Ad}^*(x_2)f &= \text{Ad}^*(x_1)(f + t_2X_d^*) = f + t_1X_d^* + t_2\text{Ad}^*(x_1)X_d^* \\ &= f + (t_1 + t_2)X_d^* + g \end{aligned}$$

for some $g \in \mathfrak{a}^\perp$. Thus

$$f + (t_1 + t_2) X_d^* \in \text{Ad}^*(G) f + \mathfrak{a}^\perp = \text{Ad}^*(G) f.$$

Suppose that $(-\varepsilon, \varepsilon) \subseteq T$ for some $\varepsilon > 0$. Then $T = \mathbb{R}$ since T is a subgroup of \mathbb{R} . However, $f + \mathbb{R} X_d^* \subseteq \text{Ad}^*(G) f$ and $\text{Ad}^*(G) f + \mathfrak{a}^\perp \subseteq \text{Ad}^*(G) f$ implies that $\text{Ad}^*(G) f + \mathfrak{b}^\perp \subseteq \text{Ad}^*(G) f$, contradicting the choice of d . Hence there exist a sequence $(t_n)_n$ in \mathbb{R} such that $t_n \rightarrow 0$ and $f + t_n X_d^* \notin \text{Ad}^*(G) f$ for all n . Then, with $f_n = f + t_n X_d^*$, all statements of (ii) except the last one are obvious.

For $g \in \mathfrak{g}^*$, let d_g denote the dimension of $\text{Ad}^*(G) g$. Since d is not a jump index for f , we have $d_f = (m - d) + \dim(\text{Ad}^*(G) f | \mathfrak{b})$. Since $f_n \rightarrow f$ in \mathfrak{g}^* , $d_{f_n} \geq d_f$ eventually. Now, $f_n | \mathfrak{b} = f | \mathfrak{b}$ and d is also not a jump index for f_n . It follows that $\text{Ad}^*(G) f_n + \mathfrak{a}^\perp \subseteq \text{Ad}^*(G) f_n$ eventually. ■

LEMMA 2. *Retain the hypotheses and notation of Lemma 1, and let $(f_n) \subseteq \mathfrak{g}^*$ be a sequence with the properties (i) and (ii) of Lemma 1. Then*

$$\text{tr } \pi_{f_n}(\varphi) \rightarrow \text{tr } \pi_f(\varphi)$$

for every C^∞ -function φ on G with compact support.

Proof. For $l \in \mathfrak{g}^*$ let \mathfrak{r}_l denote the radical of l . We first verify that if $l + \mathfrak{a}^\perp \subseteq \text{Ad}^*(G) l$, then $\mathfrak{r}_l \subseteq \mathfrak{a}$. For that, suppose there exists $X \in \mathfrak{r}_l \setminus \mathfrak{a}$ and choose $h \in \mathfrak{a}^\perp$ with $h(X) \neq 0$. By hypothesis, $l + h = \text{Ad}^*(\exp Y) l$ for some $Y \in \mathfrak{g}$. Then, since $X \in \mathfrak{r}_l$,

$$l(X) + h(X) = \text{Ad}^*(\exp Y) l(X) = e^{\text{ad}^*(Y)} l(X) = l(X),$$

a contradiction.

We apply this to $f_n = f + t_n X_d^*$. By the above, $\mathfrak{r}_{f_n} \subseteq \mathfrak{a}$. Now, since $[Y, \mathfrak{g}] \subseteq \mathfrak{b}$, we have for $Y \in \mathfrak{a}$, $Z \in \mathfrak{g}$ and $t \in \mathbb{R}$,

$$(f + t X_d^*)([Y, Z]) = f([Y, Z]) + t X_d^*([Y, Z]) = f([Y, Z]).$$

Together with $\mathfrak{r}_{f_n} \subseteq \mathfrak{a}$ this shows that $\mathfrak{r}_{f_n} = \mathfrak{r}_f$ for all n .

Now, let $\mathfrak{g} = \mathfrak{g}_m \supseteq \mathfrak{g}_{m-1} \supseteq \cdots \supseteq \mathfrak{g}_0 = \{0\}$ be a Jordan–Hölder sequence for \mathfrak{g} . For $l \in \mathfrak{g}^*$, let J_l denote the set of jump indices of l with respect to this Jordan–Hölder sequence, that is,

$$J_l = \{1 \leq j \leq m: \mathfrak{g}_j \not\subseteq \mathfrak{g}_{j-1} + \mathfrak{r}_l\}.$$

Let $E = J_f$ and $\hat{G}_E = \{\pi_l : l \in \mathfrak{g}^* \text{ such that } J_l = E\}$. Since $r_{f_n} = r_f$ for all n , we have $\pi_{f_n} \in \hat{G}_E$ for all n . By Lemma 4.4.4. of [20], the function $\pi \rightarrow \text{tr } \pi(\varphi)$ is continuous on \hat{G}_E for every $\varphi \in C_c^\infty(G)$, the algebra of C^∞ -functions with compact support on G . Thus

$$\text{tr } \pi_{f_n}(\varphi) \rightarrow \text{tr } \pi_f(\varphi).$$

for each $\varphi \in C_c^\infty(G)$. ■

Proof of Theorem 1. We first reduce to the case when G is a Lie group. G being connected, it is a projective limit of Lie groups, and π is in fact a representation of one of these Lie quotients. Thus there is a compact normal subgroup K of G such that G/K is a Lie group and $\pi \in \widehat{G/K}$. It suffices to show that the lower multiplicity of π relative to $C^*(G/K)$ is 1. Therefore we can assume that G is a Lie group.

Let H be the simply connected covering group of G , \mathfrak{h} its Lie algebra and $q: H \rightarrow G$ the covering homomorphism. Let $\rho = \pi \circ q \in \hat{H}$ and choose $f \in \mathfrak{h}^*$ so that $\rho = \pi_f$. With \mathfrak{z} the centre of \mathfrak{h} , suppose first that $\text{Ad}^*(H)f \neq f + \mathfrak{z}^\perp$. Let $(f_n)_n$ be a sequence in \mathfrak{h}^* as in Lemma 1 and set $\rho_n = \pi_{f_n}$. By Lemma 2, $\text{tr } \rho_n(\varphi) \rightarrow \text{tr } \rho(\varphi)$ for every $\varphi \in C_c^\infty(H)$. Theorem 4.1 of [4] shows that $M_L(\rho, (\rho_n)_n) = 1$. Now, the kernel K of q is contained in the centre of H , $\rho(K) = \{1\}$ and $f_n|_{\mathfrak{z}} = f|_{\mathfrak{z}}$. Thus $\rho_n(K) = \{1\}$ and hence $\rho_n = \pi_n \circ q$ where $\pi_n \in \hat{G}$, $n \in \mathbb{N}$. It follows that $\pi_n \rightarrow \pi$ in \hat{G} and

$$M_L(\pi, (\pi_n)_n) \leq M_L(\rho, (\rho_n)_n) = 1.$$

This shows that $M_L(\pi) = 1$.

Finally, suppose that $\text{Ad}^*(H)f = f + \mathfrak{z}^\perp$. Since $q(Z(H)) = Z(G)$, it follows that $\pi \sim \text{ind}_{Z(G)}^G(\pi|_{Z(G)})$. Then $Z(G)$ cannot be compact because otherwise $\{\pi\}$ was open in \hat{G} . Thus the kernel K of q is not cocompact in H , and hence there is an ideal \mathfrak{k} of codimension one in \mathfrak{z} such that $K \subseteq \exp \mathfrak{k}$. Now the set of all $l \in \mathfrak{h}^*$ such that $\text{Ad}^*(H)l = l + \mathfrak{z}^\perp$ is open in \mathfrak{h}^* . Hence there exists a sequence $(f_n)_n$ in \mathfrak{h}^* with the following properties: $f_n \in \mathfrak{k}^\perp$, $f_n \notin \text{Ad}^*(H)f$ and $\text{Ad}^*(H)f_n = f_n + \mathfrak{z}^\perp$ for all n , and $f_n \rightarrow f$ in \mathfrak{h}^* . Define $\pi_n \in \hat{G}$ by $\pi_n(xK) = \pi_{f_n}(x)$ for $x \in H$. Then $\pi_n \neq \pi$ for all n , $\pi_n \rightarrow \pi$ in \hat{G} and

$$M_L(\pi, (\pi_n)_n) \leq M_L(\pi_f, (\pi_{f_n})_n).$$

However, $M_L(\pi_f, (\pi_{f_n})_n) = 1$ since, with $m = \dim \mathfrak{h}$, $k = \dim \mathfrak{z}$ and the notation in the proof of Lemma 2, $J_{f_n} = J_f = \{k+1, \dots, m\}$ and therefore $\text{tr } \pi_{f_n}(\varphi) \rightarrow \text{tr } \pi_f(\varphi)$ for every $\varphi \in C_c^\infty(H)$ by [20, Lemma 4.4.4]. It follows that $M_L(\pi) = 1$. ■

4. PROOF OF THEOREM 2

To establish Theorem 2, we need the following technical lemma the statement of which is not surprising but appears to be unknown.

LEMMA 3. *Let G be an arbitrary locally compact group. Let N be a closed normal subgroup of G and let σ be a unitary representation of N . If the C^* -algebra $\text{ind}_N^G \sigma(C^*(G))$ is of finite dimension d , then $[G : N] \leq d$.*

Proof. Let $\pi = \text{ind}_N^G \sigma$ and suppose that N has at least $d + 1$ different cosets $N = a_0 N, \dots, a_d N$. Choose $v \in H_\sigma$ and $f \in C_c(N)$ such that $\sigma(f) v \neq 0$. For $0 \leq j \leq d$, define $\mu_j \in M(G)$, the measure algebra of G , by

$$\mu_j(\varphi) = \int_N \varphi(a_j n) f(n) dn,$$

$\varphi \in C_c(G)$. There exists an open neighbourhood V of e in G such that $a_j V N \cap a_k V N = \emptyset$ for $j \neq k$. Since $\sigma(f) v \neq 0$, there exists $g \in C_c(G)$ vanishing outside of V such that $\sigma(f * g|N) v \neq 0$. Define $\xi: G \rightarrow H_\sigma$ by

$$\xi(x) = \int_N g(xm) \sigma(m) v dm,$$

$x \in G$. Then ξ is continuous, has compact support modulo N and satisfies the covariance formula $\xi(xn) = \sigma(n^{-1}) \xi(x)$ for all $x \in G$ and $n \in N$, whence $\xi \in H_\pi$. Since $\pi(L^1(G)) = \pi(C^*(G))$ has dimension d and $\pi(L^1(G))$ is weakly dense in $\pi(M(G))$, there exist $\lambda_0, \dots, \lambda_d \in \mathbb{C}$, not all of them zero, such that

$$\sum_{j=0}^d \lambda_j \pi(\mu_j) \xi = 0.$$

Now, since g vanishes on $G \setminus V$, for $x \notin a_j V N$,

$$\begin{aligned} \pi(\mu_j) \xi(x) &= \int_N \xi((a_j n)^{-1} x) f(n) dn \\ &= \int_N f(n) \left(\int_N g((a_j n)^{-1} xm) \sigma(m) v dm \right) dn = 0, \end{aligned}$$

whereas $\pi(\mu_j) \zeta(a_j) = \sigma(f * g | N) v$. It follows that, for each k ,

$$0 = \sum_{j=0}^d \lambda_j \pi(\mu_j) \zeta(a_k) = \lambda_k \sigma(f * g | N) v,$$

whence $\lambda_k = 0$ for all k , a contradiction. ■

Proof of Theorem 2. Notice first that $\{\pi\}$ cannot be open in \hat{G} because otherwise π is a finite dimensional subrepresentation of the left regular representation of G (see [21, Theorems 2.1 and 1.7]), which is impossible since G is non-compact. Thus $M_L(\pi)$ is defined.

We claim that there is a closed normal subgroup H of G such that G/H is abelian and non-compact. For that, let $\{e\} = Z_0 \subseteq Z_1 \subseteq \dots \subseteq Z_m = G$ be the ascending central series of G , and let d be minimal such that G/Z_d is compact. Then $d \geq 1$ since G is non-compact. Then G/Z_{d-1} has a cocompact centre Z_d/Z_{d-1} and hence a relatively compact commutator subgroup [9, Corollary 1 of Theorem 4.4]. Thus there exists a closed normal subgroup H of G containing Z_{d-1} such that H/Z_{d-1} is compact and G/H is abelian. Since G/Z_{d-1} is non-compact, so is G/H .

Now let $\pi \in \hat{G}$ be finite dimensional and let

$$\Gamma = \{\gamma \in \widehat{G/H} : \pi \otimes \gamma = \pi\}.$$

Then γ is a closed subgroup of $\widehat{G/H}$, since $\{\pi\}$ is closed in \hat{G} . Thus $\Gamma = \widehat{G/N}$ for some closed subgroup N containing H . Since π is finite dimensional and $\pi \sim \text{ind}_N^G(\pi | N)$, G/N must be finite by Lemma 3. Thus N/H is non-compact and hence $\widehat{N/H}$ is non-discrete. It follows that there exists a net $(\lambda_\alpha)_\alpha$ in $\widehat{G/H}$ such that $\lambda_\alpha | N \rightarrow 1_N$ and $\lambda_\alpha | N \neq 1_N$ for all α . Let $\pi_\alpha = \pi \otimes \lambda_\alpha \in \hat{G}$, then $\pi_\alpha \rightarrow \pi$ in \hat{G} . Indeed, since

$$\pi_\alpha \sim \text{ind}_N^G(\pi | N) \otimes \lambda_\alpha = \text{ind}_N^G(\pi | N \otimes \lambda_\alpha | N) \quad \text{and} \quad \text{ind}_N^G(\pi | N) \sim \pi,$$

this follows from $\lambda_\alpha | N \rightarrow 1_N$ and the fact that inducing is continuous in Fell's subgroup representation topology [7]. Also, $\pi_\alpha \neq \pi$ for each α . In fact, if $\pi_\alpha = \pi$ then $\lambda_\alpha \in \Gamma = \widehat{G/N}$, a contradiction.

Now, for $k \in \mathbb{N}$, let \hat{G}_k denote the set of all k -dimensional $\rho \in \hat{G}$. The topology on \hat{G}_k is the weakest topology for which all the functions $\rho \rightarrow \text{tr } \rho(f)$, $f \in C^*(G)$, are continuous [6, Proposition 3.6.4]. On the set $P^1(G)$ of all normalized continuous positive definite functions on G the topology $\sigma(L^\infty(G), L^1(G))$ coincides with the topology of uniform convergence on compact subsets of G [6, Theorem 13.5.2]. Since $1/k \text{tr } \rho \in P^1(G)$ for every $\rho \in \hat{G}_k$, it follows that $\text{tr } \rho_\alpha(x) \rightarrow \text{tr } \rho(x)$ uniformly on compact subsets of G whenever $\rho_\alpha \rightarrow \rho$ in \hat{G}_k .

Finally, since $\pi_\alpha \in \hat{G}_{d_\pi}$ for all α , $\pi_\alpha \rightarrow \pi$ implies that $\text{tr } \pi_\alpha(f) \rightarrow \text{tr } \pi(f)$ for all $f \in C_c(G)$. This in turn implies that $M_L(\pi, (\pi_\alpha)_\alpha) = 1$ [4, Theorem 4.1], whence $M_L(\pi) = 1$. ■

5. PROOF OF THEOREM 3

We remind the reader that a locally compact group G is called SIN-group if it has a neighbourhood basis of the identity consisting of sets V such that $x^{-1}Vx = V$ for all $x \in G$. SIN-groups have been investigated in detail in [10].

LEMMA 4. *Let G be an SIN-group with T_1 primitive ideal space, and let $\pi \in \hat{G}$ be such that $\pi(C^*(G)) \cong \mathcal{K}(\mathcal{H}_\pi)$. Then π is finite dimensional.*

Proof. Since $\text{Prim}(C^*(G))$ is a T_1 space, $\pi(C^*(G))$ is simple and hence equal to $\mathcal{K}(\mathcal{H}_\pi)$. Moreover, G being an SIN-group, $C^*(G)$ has a central approximate identity $(f_\alpha)_\alpha$ (for instance, the characteristic functions of the sets V above). Each $\pi(f_\alpha)$ is compact and a multiple of the identity operator in \mathcal{H}_π . This of course forces π to be finite dimensional. ■

The main point in establishing Theorem 3 is to show that, for such a group G , there are no open points in \hat{G} . To prove this is considerably easier when G is second countable. This is due to the fact that a separable C^* -algebra with one point dual is known to be isomorphic to the algebra $\mathcal{K}(\mathcal{H})$ of compact operators on some Hilbert space \mathcal{H} . However, to handle non-second countable groups as well requires to employ the theory of so-called characters. We briefly introduce the necessary notation.

Let G be an SIN-group and denote by G_F the open normal subgroup of G consisting of all elements with relatively compact conjugacy classes. Fix a closed normal subgroup N of G such that $N \subseteq G_F$. Crucial to the whole theory of characters is the fact that the inner automorphisms $n \rightarrow x^{-1}nx$ of N , $x \in G$, form a relatively compact subgroup of the full automorphism group $\text{Aut}(N)$ of N [10, Theorem 0.1].

For N as above, let $K(N, G)$ be the convex set of all continuous positive definite functions ψ on N such that $\psi(e) = 1$ and $\psi(x^{-1}nx) = \psi(n)$ for all $n \in N$ and $x \in G$. $K(N, G)$ is endowed with the topology of uniform convergence on compact subsets of N . The set of extreme points of $K(N, G)$ is denoted by $E(N, G)$ and the elements in $E(N, G)$ are called G -characters of N . Of course, if N is contained in the centre of G , then $E(N, G) = \hat{N}$. When $N = G$, we simply write $E(N)$ instead of $E(N, G)$ etc.

For a continuous positive definite function ψ on G , we shall denote by ρ_ψ the cyclic representation of G arising from the GNS-construction.

LEMMA 5. *Let G be an SIN-group and let $\gamma \in E(G_F, G)$ be such that $\{\gamma\}$ is open in $E(G_F, G)$. Then there exists a compact normal subgroup K of G such that γ vanishes on $G_F \setminus K$.*

Proof. Since $\{\gamma\}$ is open in $E(G_F, G)$, there exist a compact subset M of G_F and $\varepsilon > 0$ such that the neighbourhood

$$U(\gamma, M, \varepsilon) = \{\sigma \in K(G_F, G) : |\sigma(x) - \gamma(x)| < \varepsilon \text{ for all } x \in M\}$$

of γ equals $\{\gamma\}$. Because every compact subset of G is contained in some G -invariant compact set, we can assume that, in addition, M has non-void interior and is G -invariant. Let H be the subgroup generated by M . Then H is open and normal in G , and $\delta = \gamma|_H$ belongs to $E(H, G)$ [18, Proposition 2.9]. We claim that γ vanishes on $G_F \setminus H$. To verify this, consider the trivial extension of δ to G_F , that is, the function $\tilde{\delta}$ defined by $\tilde{\delta}(x) = \delta(x)$ for $x \in H$ and $\tilde{\delta}(x) = 0$ for $x \in G_F \setminus H$. Then $\tilde{\delta}$ can be approximated by convex linear combinations of elements in $E(G_F, G)$ extending δ . In fact, this follows from the Krein–Milman theorem applied to the compact convex set $\{\chi \in K(G_F, G) : \chi|_H = \delta\}$. Now, every $\tau \in E(G_F, G)$ extending δ belongs to $U(\gamma, M, \varepsilon)$ and hence equals γ . It follows that $\gamma = \tilde{\delta}$. Notice also that $\{\delta\}$ is open in $E(H, G)$.

There is a continuous surjection $s: E(H) \rightarrow E(H, G)$ defined by

$$s(\varphi)(x) = \int_B \varphi(\beta(x)) d\beta,$$

for all $x \in H$ and $\varphi \in E(H)$, where $d\beta$ is normalized Haar measure on the compact automorphism group $B = \overline{I(H, G)} \subseteq \text{Aut}(H)$ [18, Theorem 5.8]. Then $s^{-1}(\delta)$ is open in $E(H)$ and, for each $\varphi \in s^{-1}(\delta)$,

$$s^{-1}(\delta) = \{\varphi \circ \beta : \beta \in B\}$$

[18, Proposition 5.7]. Recall that an element x of a topological group is called compact if the closed subgroup generated by x is compact. Let K denote the set of all compact elements of H . Since H is a compactly generated group with relatively compact conjugacy classes, it turns out that K is a compact normal subgroup of G , and H/K is abelian and has no non-trivial compact element [9, Theorem 3.20].

This last fact implies that the dual group $\widehat{H/K}$ is connected [11, Theorem 24.17]. Moreover, $\widehat{H/K}$ acts on $E(H)$ by pointwise multiplication, and the mapping

$$\widehat{H/K} \times E(H) \rightarrow E(H), \quad (\alpha, \varphi) \rightarrow \alpha\varphi$$

is continuous. Thus, for each $\varphi \in s^{-1}(\delta)$, the subset $s(\widehat{H/K} \cdot \varphi)$ of $E(H, G)$ is connected and contains δ , so that $s(\widehat{H/K} \cdot \varphi) = \{\delta\}$.

We show next that K is open in H . By [10, Theorem 2.13 and Theorem 3.20], H is a direct product $H = V \times L$ where V is a vector group and L contains a compact open subgroup. Then $E(H) = \hat{V} \times E(L)$, and the projection p of $E(H)$ onto \hat{V} is continuous and open. Thus $p(s^{-1}(\delta))$ is a compact open subset of \hat{V} . This is impossible whenever V is non-trivial. Since L contains a compact open subgroup, we conclude that K is open in H .

Now, fix $\varphi \in s^{-1}(\delta)$. Then, as we have seen above,

$$\delta(x) = s(\alpha\varphi)(x) = \int_B \alpha(\beta(x)) \varphi(\beta(x)) d\beta$$

for all $x \in H$ and $\alpha \in \widehat{H/K}$. Let $d\alpha$ denote the normalized Haar measure on the compact abelian group $\widehat{H/K}$. Then, for all $x \in H$,

$$\delta(x) = \int_B \varphi(\beta(x)) \left(\int_{\widehat{H/K}} \alpha(\beta(x)) d\alpha \right) d\beta.$$

Finally, recall that

$$\int_{\widehat{H/K}} \alpha(y) d\alpha = 0$$

for every $y \in H \setminus K$ [11, Lemma 23.19]. It follows that, if $x \in H \setminus K$ and hence $\beta(x) \in H \setminus K$ for all $\beta \in B$, then $\delta(x) = 0$. This finishes the proof of the lemma. ■

There are various characterizations of amenability of a locally compact group (see [8]). In terms of representation theory, one of the equivalent conditions is that every irreducible representation is weakly contained in the left regular representation [8, Theorem 3.5.2].

LEMMA 6. *Let G be a non-compact amenable SIN-group with T_1 primitive ideal space. Then \hat{G} has no open points.*

Proof. Suppose that for some $\pi \in \hat{G}$ the singleton $\{\pi\}$ is open in \hat{G} . Since the primitive ideal space $\text{Prim}(C^*(G))$ is a T_1 space, $\{\pi\}$ is also closed in \hat{G} . In particular, the C^* -kernel of π is a separated point of $\text{Prim}(C^*(G))$, and hence there exists $\gamma \in E(G_F, G)$ such that $\pi \sim \text{ind}_{G_F}^G \rho_\gamma$ [13, Theorem 3.6]. Now, for any amenable SIN-group G , there is an open (and continuous) mapping $r: \hat{G} \rightarrow E(G_F, G)$ such that $\pi|_{G_F} \sim \rho_{r(\pi)}$ (see [12, Sect. 2]). Thus $\gamma = r(\pi)$, which is an isolated point of $E(G_F, G)$. By

Lemma 5 there exists a compact normal subgroup K of G such that γ vanishes on $G_F \backslash K$. Since

$$\text{ind}_K^{G_F}(\rho_\gamma | K) \sim \text{ind}_K^{G_F} \rho_{\gamma|K} = \rho_{\gamma|K} \widetilde{\phantom{\rho_{\gamma|K}}},$$

it follows that $\rho_\gamma \sim \text{ind}_K^{G_F}(\rho_\gamma | K)$. Hence, since $\rho_\gamma | K \sim G(\sigma)$ for some $\sigma \in \hat{K}$,

$$\pi \sim \text{ind}_{G_F}^G \rho_\gamma \sim \text{ind}_{G_F}^G (\text{ind}_K^{G_F}(\rho_\gamma | K)) = \text{ind}_K^G(\rho_\gamma | K) \sim \text{ind}_K^G \sigma.$$

Thus $\text{ind}_K^G \sigma$ is a multiple of π because $\{\pi\}$ is closed in \hat{G} . In particular, $\text{ind}_K^G \sigma$ is a type I representation [6, Proposition 5.4.7]. Moreover, K being compact, σ is a subrepresentation of the left regular representation λ_K of K . Hence $\text{ind}_K^G \sigma$ is a subrepresentation of $\text{ind}_K^G \lambda_K$, which is (unitarily equivalent to) the left regular representation λ_G of G . Now, since G is an SIN-group, the von Neumann algebra $VN(G)$ generated by λ_G is finite [6, Proposition 13.10.5]. Thus π is an irreducible representation of a type I finite von Neumann algebra and hence has to be finite dimensional. Finally, since $\pi \sim \text{ind}_K^G \sigma$, Lemma 3 shows that G/K is finite. This proves that G is compact, which contradicts the hypothesis. Thus there are no open points in \hat{G} . ■

Now the proofs of Theorem 3 and of the Corollary follow quickly.

Proof of Theorem 3. By Lemma 6, there are no open points in \hat{G} and hence $M_L(\pi)$ is defined for every $\pi \in \hat{G}$.

Suppose that $M_L(\pi) < \infty$. Then, by Theorem 4.4 of [1], $\pi(C^*(G)) \cong \mathcal{K}(\mathcal{H}_\pi)$ since $\{\pi\}$ is not open in \hat{G} . Lemma 4 yields that π is finite dimensional. ■

Proof of Corollary. We know from Theorem 2 that if $d_\pi < \infty$, then $M_L(\pi) = 1$. To prove the corollary, it therefore suffices to show that conversely $M_L(\pi) < \infty$ implies that $d_\pi < \infty$. For that, recall first that every nilpotent group is amenable (see [8]). Next, notice that every SIN-group is a projective limit of Lie groups [17, Lemma 4.3]. In particular, there exists a compact normal subgroup C of G such that G/C is a Lie group. This property together with the fact that G is nilpotent guarantees that $\text{Prim}(C^*(G))$ is a T_1 space [14]. The statement now follows from Theorem 3. ■

We conclude the paper with two remarks concerning Lemma 6.

Remark 1. Lemma 6 does not remain true if \hat{G} is replaced by $\text{Prim}(C^*(G))$. In fact, there exist non-compact nilpotent SIN-groups the

primitive ideal space of which has open points. To see this, let $G = \mathbb{Z} \times \mathbb{Z} \times \mathbb{T}$ with the product topology and multiplication given by

$$(m, n, z)(p, q, w) = (m + p, n + q, zwe^{imq}),$$

$m, n, p, q \in \mathbb{Z}$, $z, w \in \mathbb{T}$. Then G is a 2-step nilpotent group with open centre $Z = \{0\} \times \{0\} \times \mathbb{T}$. The mapping $\alpha \rightarrow \ker \pi_\alpha$ is a homeomorphism between $E(G)$ and $\text{Prim}(C^*(G))$ [18, Theorem 5.2]. Let $\alpha \in E(G)$ and $\xi \in H_{\pi_\alpha}$ such that $\alpha(x) = \langle \rho_\alpha(x) \xi, \xi \rangle$ for all $x \in G$. Since ρ_α is factorial and $[G, G] \subseteq Z$, we have for all $x, y \in G$

$$\begin{aligned} \alpha(x) &= \alpha(yxy^{-1}) = \alpha([y, x]x) = \langle \rho_\alpha([y, x]) \rho_\alpha(x) \xi, \xi \rangle \\ &= \alpha([y, x]) \langle \rho_\alpha(x) \xi, \xi \rangle = \alpha([y, x]) \alpha(x). \end{aligned}$$

Now it is easily checked that, for every $x \in G \setminus Z$, $[G, x] = \{[y, x] : y \in G\}$ is a dense subgroup of Z . It follows that if the restriction of α to Z is non-trivial and $x \in G \setminus Z$, then $\alpha([y, x]) \neq 1$ for some $y \in G$, and hence $\alpha(x) = 0$. Thus

$$E(G) = \widehat{G/Z} \cup \{\tilde{\gamma} : \gamma \in \hat{Z}, \gamma \neq 1_Z\}.$$

Clearly, each such $\tilde{\gamma}$ is an open point of $E(G)$, and hence the singleton $\{\ker \pi_{\tilde{\gamma}}\}$ is open in $\text{Prim}(C^*(G))$.

Remark 2. Also, Lemma 6 does not remain true if the hypothesis that G be an SIN-group is weakened to the effect that G is only supposed to be an IN-group (that is, G has at least one compact invariant neighbourhood of the identity). An example is provided by the Weyl–Heisenberg group W , the quotient of the real Heisenberg group modulo the central integer subgroup. It is well known that every infinite dimensional irreducible representation of W (for example, the Schrödinger representation) forms an open point of \hat{W} .

In addition, this example also shows that Lemma 4 does not generalize to IN-groups.

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